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PLASTIC RESPONSE OF ORTHOTROPIC CIRCULAR PLATES UNDER BLAST LOADING

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Abstract—The dynamical behaviour of a simply supported, orthotropic, circular plate subjected to strong blast is considered. The blast is assumed to impart an axisymmetric transverse, velocity which has a general Gaussian distribution spatially. It is concluded that the rate of growth of plastic regimes and the final plastic deformation strongly depend upon the initial Gaussian distribution parameter.

NOTATION

b	radius of circular plate
a	$(1/\sqrt{(2)}S)$, S being standard deviation
c	$a.b$
$\dot{K}_r$	radial curvature rate
$\dot{K}_\theta$	circumferential curvature rate
M_{r0}	resultant yield moment in radial direction
$M_{\theta 0}$	resultant yield moment in circumferential direction
K	$(M_{\theta 0}/M_{r0})$
M_r	resultant radial bending moment
M_θ	resultant circumferential bending moment
r, θ	radial, circumferential plate co-ordinates
t	time
t^*	time of cessation of all plate motion
t_1	time at which hinge circle radius has decreased to zero
$W(r, t)$	plate deflection in transverse direction
$\dot{W}(r, t)$	plate velocity
W^*	permanent plate deflection
V_0	$\dot{W}(0, 0)$
$\delta(t)$	Dirac δ-function
μ	mass per unit area of plate material'
$\rho(t)$	radius of hinge circle
ρ_0	initial position of hinge circle at $t = 0$
ρ_1	position of hinge circle at $t = t_1$

1. INTRODUCTION

Theoretical and experimental investigations into inelastic response of engineering structures under high intensity short duration loads are necessary in order to permit reliable predictions of structural damage, to automobiles, aircraft, spacecraft and high speed marine craft which is sustained during collisions, the safety of pressure vessels which contain nuclear reactors and to develop energy absorbing devices for various applications. Dynamical bending of perfectly plastic, thin circular plates has received considerable attention in the last two decades. A simple formula for permanent central deflection of a simply supported circular, isotropic, rigid-plastic plate caused by a uniformly distributed impulse was obtained by Wang[1] using the theory of Hopkins and Prager[2]. The influence of different boundary conditions and various axisymmetric dynamic loads with arbitrary time history on the behaviour of perfectly plastic, circular plates has been further studied by Perzyna[3], Shapiro[4], Florence[5], Thomson[6] and Conroy[7]. Other investigations taking into account the strengthening effects due to strain-rate sensibility and membrane forces include those of Perrone[8], Florence[9], Duffey[10], Wierzbicki[11] and Jones[12]. Mazalov and Nemirovskii[13] have investigated the dynamical behaviour of piece-wise non-homogeneous plates under uniformly distributed pressure pulse. Numerous references for works done in this field can be found in survey paper by Jones *et al.*[14].

It is clear from the above review that most of the studies so far have been concentrated on the dynamic deformation of isotropic circular plates under axisymmetric pressure pulse.

The present paper is concerned with the response of a simply supported, orthotropic circular plate under a strong blast. The blast is assumed to impart a transverse velocity which is axisymmetric with a general Gaussian distribution spatially. The material of the plate exhibits polar orthotropy flowing according to modified Tresca yield condition and associated flow rules [15]. Only the bending action of the plate is considered. The effects of geometry changes are neglected.

2. FUNDAMENTAL EQUATIONS

Consider a thin circular plate simply supported along its edge and subjected to a strong blast at time $t = 0$. The blast imparts a velocity which has a Gaussian distribution (Fig. 1) given by

$$\dot{W} = V_0 e^{-a^2 r^2} \delta(t). \tag{1}$$

The dynamic equilibrium equations governing the rigid plastic behaviour of the plate is

$$\frac{d}{dr}(rM_r) = M_\theta + \int_0^r \mu \ddot{W} r \, dr \tag{2}$$

subjected to the boundary conditions

$$\left.\begin{aligned} M_r(b, t) &= 0 \\ \dot{W}(b, t) &= 0. \end{aligned}\right\} \tag{3}$$

Two different cases of orthotropy are of practical interest.

(1) The resultant yield moment in the radial direction being greater than that in the circumferential direction. ($M_{r0} > M_{\theta 0}$)

(2) The resultant yield moment in the circumferential direction being greater than that in the radial direction. ($M_{\theta 0} > M_{r0}$)

The yield conditions and associated flow rules for the above cases of orthotropy are (Figs. 2, 3).

Case 1. ($M_{r0} > M_{\theta 0}$)

Regime A	*Regime AB*
$M_r = M_\theta = M_{\theta 0}$	$0 \leqslant M_r \leqslant M_{\theta 0}$
$\dot{K}_r = 0$	$\dot{K}_r = 0$
$\dot{K}_\theta \geqslant 0$	$\dot{K}_\theta \geqslant 0$

Case 2. ($M_{\theta\theta} > M_{r0}$)

Regime BC

$0 < M_r < M_{r0}$

$M_\theta = M_{\theta 0}$

$\dot{K}_r = 0$

$\dot{K}_\theta \geqslant 0$

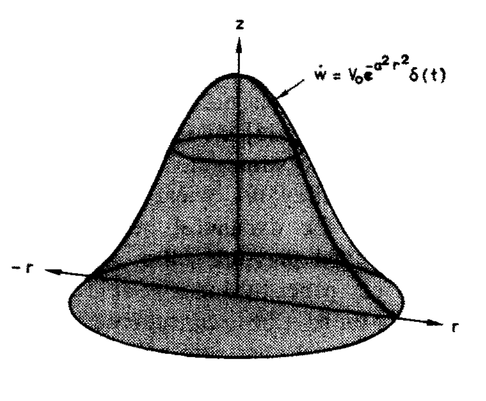

Fig. 1. Gaussian velocity distribution.

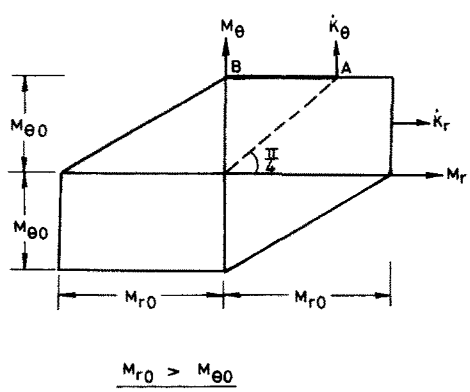

Fig. 2. Modified Tresca yield condition (($M_{r0} > M_{\theta 0}$).

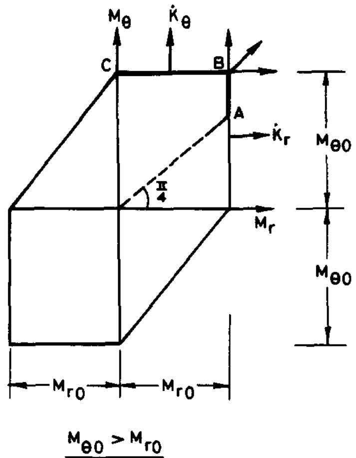

Fig. 3. Modified Tresca yied condition ($M_{\theta 0} > M_{r0}$).

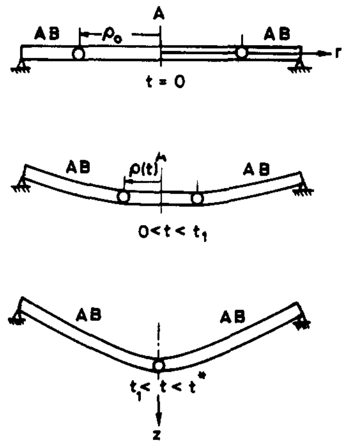

Fig. 4. Modes of deformation.

3. MODES OF DEFORMATION

Case 1. ($M_{r0} > M_{\theta 0}$)

When the blast is imparted to the plate at time $t = 0$, the plate gets divided into two different plasticity regimes A and AB (Fig. 2) with a hinge circle located at the radius ρ_0 (Fig. 4). For time $t > 0$, the hinge circle starts moving towards the centre of the plate. At time $t = t_1$ the hinge circle shrinks to the centre of the plate. The plate continues to deform in the form of an inverted cone until all the kinetic energy imparted to the plate by the blast is completely dissipated by plastic deformation and the plate comes to rest at time $t = t^*$.

Case 2. ($M_{\theta 0} > M_{r0}$)

The first phase of deformation of this type of plate is similar to the one discussed in Case 1 except that at time $t = t_1$ the hinge circle has radius ρ_1 given by the following equation

$$2\left(\frac{\rho_1}{b}\right)^3(K-1) - 3\left(\frac{\rho_1}{b}\right)^2 + K - 1 = 0. \tag{4}$$

The plate continues to deform in the form of a truncated cone with an inner rigid portion of radius ρ_1. At time $t = t^*$ the velocity of the plate becomes zero everywhere and the plate comes to rest giving a permanent deformed shape.

4. SOLUTION

Case 1. ($M_{r0} > M_{\theta 0}$)

The solution is obtained separately for the two phases of deformation described above.

Phase 1. ($0 \leq t \leq t_1$)

The velocity profile satisfying the flow rule, boundary conditions and appropriate continuity and discontinuity conditions [2] is

$$\dot{W} = V_0 e^{-a^2\rho^2} \quad \text{for} \quad 0 \leq r \leq \rho(t) \tag{5(a)}$$

$$\dot{W} = V_0 e^{-a^2\rho^2} \frac{b-r}{b-\rho(t)} \quad \rho(t) \leq r \leq b \tag{5(b)}$$

Using eqn 5(b) and (2) with (3) the differential equation for the motion of hinge circle is obtained as

$$\frac{12}{\mu V_0}\frac{M_{\theta 0}}{b^2}\,dt = \frac{e^{-a^2\rho^2}}{b}(1 - 2a^2b + 2a^2\rho^2)\left[-1 - 2\frac{\rho}{b} + 3\left(\frac{\rho}{b}\right)^2\right]d\rho. \tag{6}$$

The solution of eqn (6) gives the position of hinge circle at any time "t". In a non-dimensional form it can be expressed as

$$\frac{t}{t_1}=\frac{S_1'+K_1'S_2'+K_2'}{K_3'+K_2'} \tag{7}$$

where

$$S_1'=\left[-1+\frac{6}{c^2}\left(1-\frac{\rho}{b}\right)-\frac{\rho}{b}+5\left(\frac{\rho}{b}\right)^2-3\left(\frac{\rho}{b}\right)^3\right]e^{-c^2(\rho/b)^2}$$

$$S_2'=\mathrm{erf}\left(c\cdot\frac{\rho}{b}\right)$$

$$K_1'=\frac{3\sqrt{\pi}}{c^3}$$

$$K_2'=\left[1-\frac{6}{c^2}\left(1-\frac{\rho_0}{b}\right)+\frac{\rho_0}{b}-5\left(\frac{\rho_0}{b}\right)^2+3\left(\frac{\rho_0}{b}\right)^3\right]e^{-c^2(\rho_0/b)^2}-K_1'\,\mathrm{erf}\left(c\cdot\frac{\rho_0}{b}\right) \tag{7a}$$

$$K_3'=-1+\frac{6}{c^2}$$

ρ_0/b in eqn (7a) is the position of the hinge circle at time $t=0$. Since the kinetic energy imparted to the plate is maximum at $t=0$ and decreases thereafter, $(\mathrm{d}\rho/\mathrm{d}t)$ must be negative. Using this condition in eqn (7) we get

$$\frac{\rho_0}{b}=\frac{1-\sqrt{(1-2/(c)^2)}}{2}. \tag{8}$$

This equation is valid for Gaussian distribution parameter $c\geqslant\sqrt{2}$. When $c=\sqrt{(2)}$, $\rho_0/b=0{\cdot}5$. For a blast which imparts a uniform velocity to the entire plate c tends to zero and the hinge circle starts at the support [2]. The values of ρ_0/b for $0<c<\sqrt{2}$ can be interpolated between 1 and 0·5.

The velocity field given by eqn (5) is integrated using eqn (6) and the condition that at $r=\rho$, $M_r=M_\theta=M_{\theta 0}$ to get the deflection of the plate up to the time $t=t_1$ as

$$\frac{12M_{\theta 0}}{\mu V_0^2b^2}W(r,t_1)=\left(1-\frac{r}{b}\right)\left[\left\{\frac{3}{2(c)^2}-\frac{1}{2}\right\}-\left\{\frac{3}{2}\left(\frac{\rho_0}{b}\right)^2+\frac{3}{2c^2}-\frac{1}{2}-\frac{\rho_0}{b}\right\}\right]e^{-2c^2(\rho_0/b)^2}$$

$$\text{for}\quad \frac{\rho_0}{b}\leqslant r\leqslant 1 \tag{9}$$

$$\frac{12M_{\theta 0}}{\mu V_0^2b^2}W(r,t_1)=\frac{12M_{\theta 0}}{\mu V_0b^2}t_r+\left(1-\frac{r}{b}\right)\left[\frac{3}{2c^2}-\frac{1}{2}-\left\{\frac{3}{2}\left(\frac{r}{b}\right)^2+\frac{3}{2c^2}-\frac{1}{2}-\frac{r}{b}\right\}\right]e^{-2c^2(r/b)^2}$$

$$\text{for}\quad \frac{\rho(t)}{b}\leqslant\frac{r}{b}\leqslant\frac{\rho_0}{b} \tag{10}$$

$$\frac{12M_{\theta^0}}{\mu V_0^2b^2}W(r,t_1)=\frac{12M_{\theta^0}}{\mu V_0b^2}e^{-c^2(r/b)^2}t_1\qquad \text{for}\quad 0\leqslant\frac{r}{b}\leqslant\frac{\rho(t)}{b} \tag{11}$$

where t_r is the time when the hinge circle is located at a radius "r" which can be readily obtained from the eqn (6).

Phase 2. $(t_1\leqslant t\leqslant t^*)$

The cumulative deflection of the plate in this phase can be written as

$$W(r,t)=W(r,t_1)+\phi(t)\left(1-\frac{r}{b}\right) \tag{12}$$

where $W(r, t_1)$ given by eqns (9)–(11). The arbitrary function $\phi(t)$ is determined using eqn (2) with the conditions

$$
\begin{array}{lll}
M_r = M_{\theta 0} & \text{at} & r=0 \\
M_r = 0 & \text{at} & r=b \\
\phi(t_1)=0 & & \\
\dot{\phi}(t_1)=V_0. & &
\end{array}
$$

Equation (12) thus reduces to

$$W(r,t)=W(r,t_1)+(1-r/b)\left[\frac{-6M_{\theta 0}}{b^2}(t-t_1)^2+V_0(t-t_1)\right]. \tag{13}$$

At time $t = t^*$ the plate comes to rest and velocity of the plate is zero. With this condition we get

$$\frac{12M_{\theta 0}}{\mu V_0 b^2}t^*=\frac{12M_{\theta 0}}{\mu V_0 b^2}t_1+1.$$

The permanent deflection of the plate at $t = t^*$ is then given by

$$\frac{12M_{\theta 0}}{\mu V_0^2 b^2}W^*(r,t^*)=\frac{12M_{\theta 0}}{\mu V_0^2 b^2}W(r,t_1)+\frac{1}{2}(1-r/b). \tag{14}$$

Case 2. ($M_{\theta 0} > M_{r0}$)

As described earlier, the deformation of the plate is characterized by a plastically flowing annulus surrounding a central rigid portion of the plate. The plastic flow of the annular region is governed by the regime BC of the yield diagram (Fig. 3). At the centre of the plate $M_r = M_\theta = M_{r0}$ because of symmetry. At the boundary of the rigid region $r = \rho_1$, $M_r = M_{r0}$ and $M_\theta = M_{\theta 0}$. Proceeding on the lines similar to the previous case we get the differential equation for the motion of the hinge circle as

$$\begin{aligned}\frac{12M_{\theta 0}}{\mu V_0 b^2}\,\mathrm{d}t=\frac{\mathrm{e}^{-c^2(\rho/b)^2}}{b}&\left[1-K_1\left(\frac{\rho}{b}\right)+K_1^2\left(\frac{\rho}{b}\right)^2-K_1^3\left(\frac{\rho}{b}\right)^3\right]\left[-1+2c^2\left(\frac{\rho}{b}\right)+2c^2\left(\frac{\rho}{b}\right)^2\right.\\&\left.-2\left(\frac{\rho}{b}\right)-10c^2\left(\frac{\rho}{b}\right)^3+3\left(\frac{\rho}{b}\right)^2+6c^2\left(\frac{\rho}{b}\right)^4\right]\mathrm{d}\rho\end{aligned} \tag{15}$$

where

$$K_1=\frac{1-K}{K}.$$

Equation (15) is solved to get the position of hinge circle at any time "t" as,

$$\frac{t}{t_1}=\frac{S_1A_1+A_2-S_1A_3-A_4}{S_1A_5+A_6-S_1A_3-A_4} \tag{16}$$

where

$$S_1=\frac{3}{c^2}-\frac{K_1}{2c}+\frac{17K_1}{4c^3}+\frac{K_1^2}{2c^3}+\frac{54K_1^2}{8c^5}-\frac{3K_1^3}{4c^3}+\frac{81K_1^3}{8c^5}$$

$$A_1=S_2'$$

$$A_2=\mathrm{e}^{-c^2(\rho/b)^2}\left[S_2+S_3\left(\frac{\rho}{b}\right)+S_4\left(\frac{\rho}{b}\right)^2+S_5\left(\frac{\rho}{b}\right)^3+S_6\left(\frac{\rho}{b}\right)^4+S_7\left(\frac{\rho}{b}\right)^5+S_8\left(\frac{\rho}{b}\right)^6\right]$$

$$S_2 = -1 + \frac{6}{c^2} + \frac{K_1}{2c^2} + \frac{15K_1}{2c^4} - \frac{K_1^2}{c^2} + \frac{11K_1^2}{c^4} + \frac{3K_1^3}{2c^4} + \frac{21K_1^2}{c^6}$$

$$S_3 = -1 - \frac{6}{(c)^2} + K_1 - \frac{17K_1}{2(c)^2} - \frac{K_1^2}{(c)^2} - \frac{54K_1^2}{4(c)^4} + \frac{3K_1^3}{2(c)^2} - \frac{81K_1^3}{4(c)^4}$$

$$S_4 = 5 + K_1 + \frac{15K_1}{2(c)^2} - K_1^2 + \frac{11K_1^2}{(c)^2} + \frac{3K_1^3}{2(c)^2} + \frac{21K_1^3}{(c)^4}$$

$$S_5 = -3 - 5K_1 - K_1^2 - \frac{9K_1^2}{(c)^2} + 2K_1^3 - \frac{27K_1^3}{2(c)^2}$$

$$S_6 = 3K_1 + 5K_1^2 + \frac{21K_1^3}{2(c)^2}$$

$$S_7 = 3K_1^2 - 5K_1^3$$

$$S_8 = 3K_1^3$$

$$A_3 = \operatorname{erf}(c\rho_0/b)$$

$$A_4 = e^{-(c)^2(\rho_0/b)^2} S_2 + S_3(\rho_0/b) + S_4(\rho_0/b)^2 + S_5(\rho_0/b)^3 + S_6(\rho_0/b)^4 + S_7(\rho_0/b)^5 + S_8(\rho_0/b)^6$$

$$A_5 = \operatorname{erf}(c\rho_1/b)$$

$$A_6 = e^{-c^2(\rho_1/b)^2}\left[S_2 + S_3\frac{\rho_1}{b} + S_4\left(\frac{\rho_1}{b}\right)^2 + S_5\left(\frac{\rho_1}{b}\right)^3 + S_6\left(\frac{\rho_1}{b}\right)^4 + S_7\left(\frac{\rho_1}{b}\right)^5 + S_8\left(\frac{\rho_1}{b}\right)^6\right].$$

The analytical method of obtaining the deflections was found to be very complex for this case. Hence a numerical method of integrating the velocity field from time $t = 0$ to $t = t^*$ was adopted. The time t^* at which the plate comes to rest is derived on the same lines as in Case 1 to get,

$$\frac{12M_{\theta 0}}{\mu V_0 b^2}t^* = \frac{1 - 6(\rho_1/b)^2 + 8(\rho_1/b)^3 - 3(\rho_1/b)^4}{(1-\rho_1/b)[1 + K_1(\rho_1/b)]}$$

where ρ_1 is determined from eqn (4).

NUMERICAL RESULTS AND CONCLUSIONS

The results of the analysis are presented in Figs. 5–9, in which the response of the plate for different load distribution parameters as well as different degrees of orthotropy are compared. The description of the spatial distribution of the blast as Gaussian facilitates the analysis to

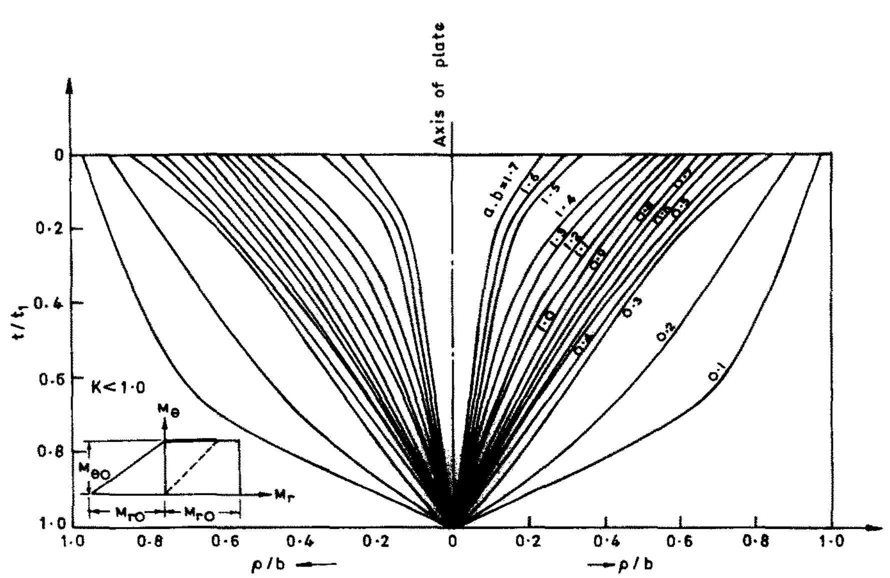

Fig. 5. Movement of hinges.

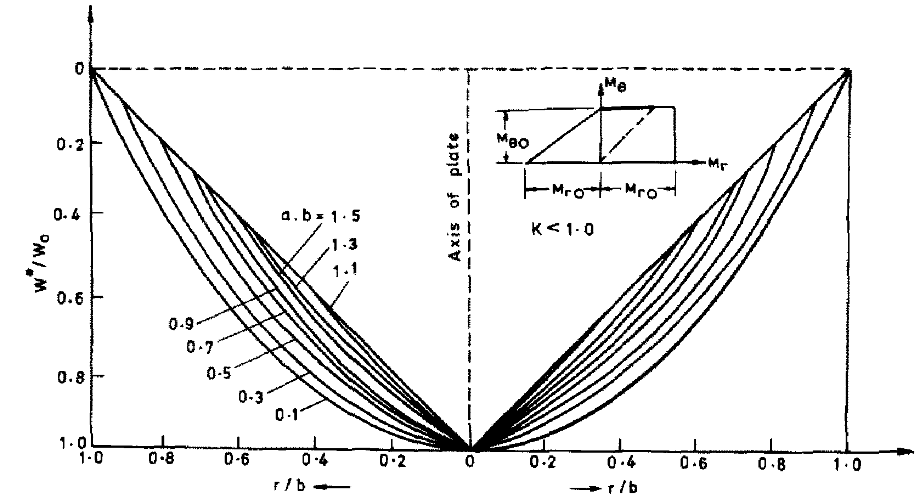

Fig. 6. Deflection of the plate.

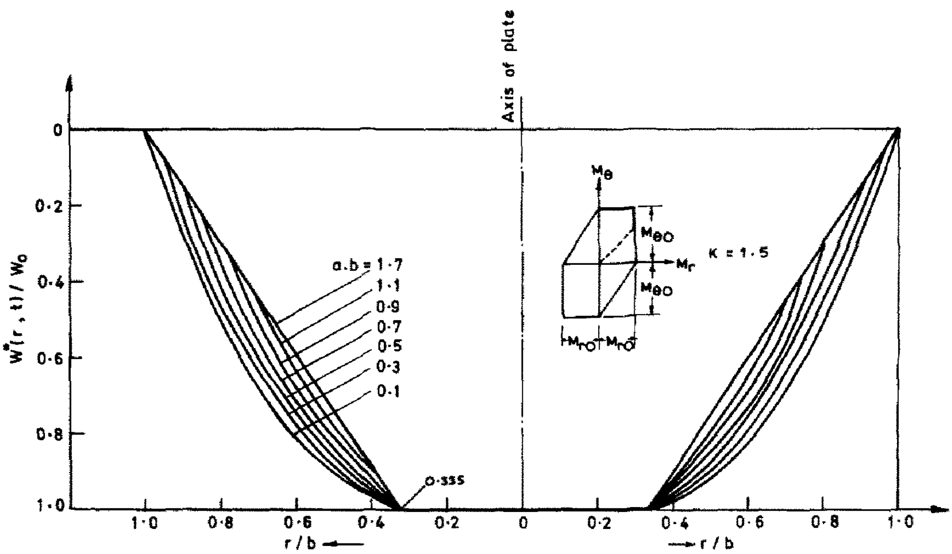

Fig. 7. Movement of hinges.

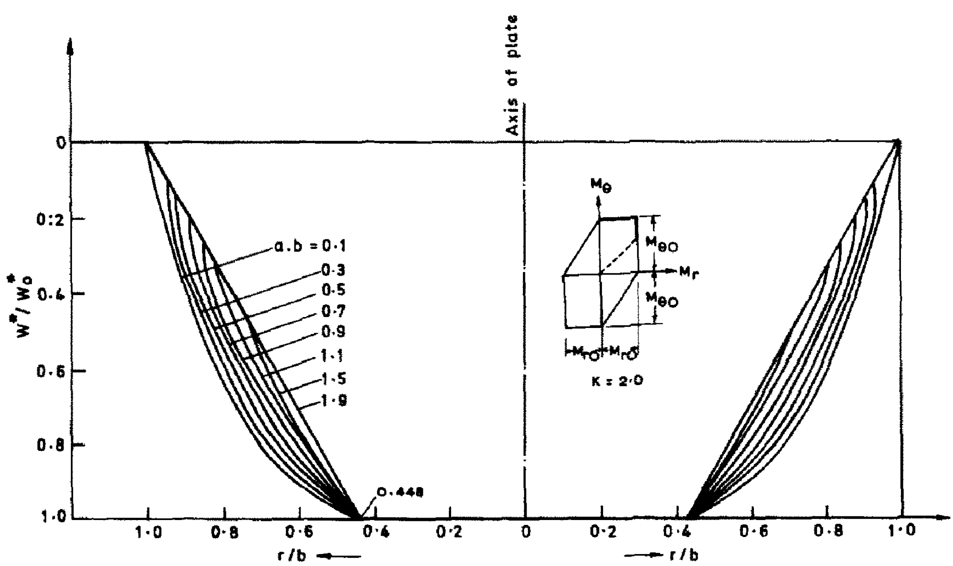

Fig. 8. Deflection of the plate.

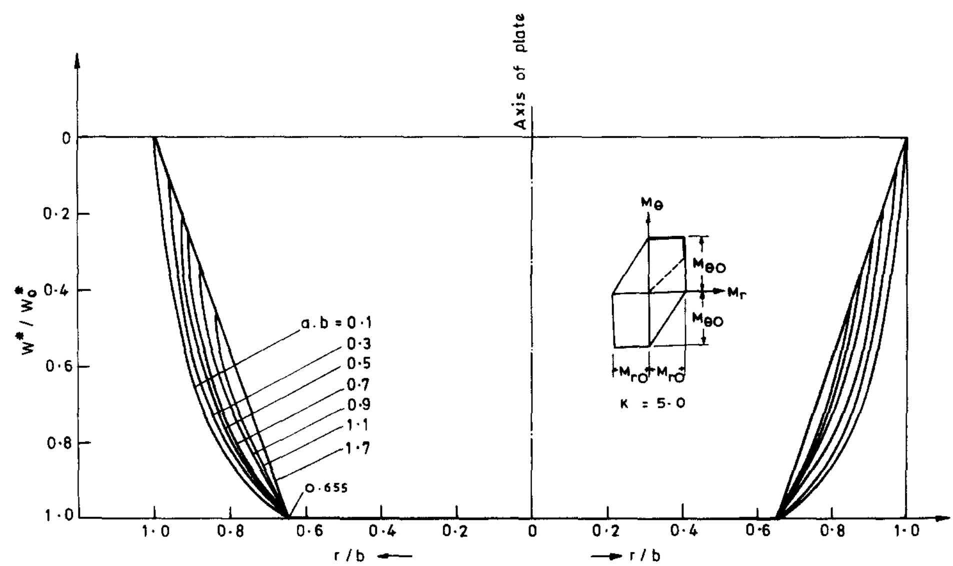

Fig. 9. Movement of hinges.

account for a variety of load distributions ranging from a pulse concentrated at the centre of the plate to a uniformly distributed pulse over the entire plate. The choice of the directions of orthotropy to be in the radial and tangential directions has considerably simplified the analysis. However, it is easy to visualize this kind of anisotropy to be a consequence of some mechanical fabrication technique; it is also possible to realise a similar situation in the case of plates with appropriately spaced radial and circumferential stiffners. The behaviour of an orthotropic plate with $M_{r0} > M_{\theta 0}$ is similar to that of an isotropic plate. For $c = 0{\cdot}1$ which approximately corresponds to a uniformly distributed impulse the deflection profile compares well with that of Wang[1]. For $c = 1{\cdot}5$ to $1{\cdot}7$ which simulates a concentrated impulse at the centre the plate deforms to the shape of an inverted cone if $M_{r0} > M_{\theta 0}$.

If the plate is stronger in the circumferential direction than in the radial direction (i.e. $M_{\theta 0} > M_{r0}$) the deformed shape of the plate is found to be of the form of a frustrum of a cone with a rigid flat portion at the centre. The exactness of the solution can be verified from the fact that it is possible to construct at least one equilibrium stress field which does not violate the yield condition in the central rigid region, e.g.

$$M_r = M_{r0}\left[1 - \frac{r}{b}\left(\frac{\rho}{b} - \frac{r}{b}\right)\right], \qquad M_\theta = M_{r0}\left[1 - \frac{1}{3}(\rho/b)^2\right].$$

It may also be noted that all the relevant continuity conditions across moving and stationary hinges are satisfied during all phases of motion of the plate.

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